UDC 531.36

# ON THE STABILITY OF STEADY MOTIONS OF CHAPLYGIN'S NONHOLONOMIC SYSTEMS

PMM Vol. 42, No. 5, 1978, pp 801-807

A. V. KARAPETIAN
(Moscow)

(Received November 14, 1977)

Stability of steady motions of Chaplygin's nonholonomic systems subjected to the action of potential and dissipative forces and possessing ignorable coordinates is investigated. A survey of available results in this field appears in [1].

1. Let us consider a scleronomous nonholonomic system subjected to forces admitting a force function. We denote the generalized coordinates by $q_1, \ldots, q_n$, and assume that the generalized velocities $q_1^{\cdot}, \ldots, q_n^{\cdot}$ are linked by $n-m$ nonintegrable relationships of the form

$$q_\mu^{\cdot} = \sum_{r=1}^{m} b_{\mu r}(q) q_r^{\cdot} \quad (\mu = m+1, \ldots, n) \tag{1.1}$$

Assuming that the system can be subjected to dissipative forces, derivatives of the Rayleigh function $F$ whose coefficients are independent of $q_\mu$, and that the kinetic energy $T$, the force function $U$, and the coefficients of links $b_{\mu r}$ are also independent of $q_\mu$, we represent the equations of motion of the system in Chaplygin's form

$$\frac{d}{dt}\frac{\partial T^*}{\partial q_r^{\cdot}} = \frac{\partial (T^* + U)}{\partial q_r} + \sum_{p,\, s=1}^{m} q_s^{\cdot} q_p^{\cdot} \sum_{\mu=m+1}^{n} \theta_{\mu p} \nu_{\mu r s} - \frac{\partial F^*}{\partial q_r^{\cdot}} \quad (r = 1, \ldots, m) \tag{1.2}$$

$$\left(\nu_{\mu r s} = \frac{\partial b_{\mu r}}{\partial q_s} - \frac{\partial b_{\mu s}}{\partial q_r}\right)$$

where

$$2T^* = \sum_{r,\, s=1}^{m} a_{rs} q_r^{\cdot} q_s^{\cdot}, \quad 2F^* = \sum_{r,\, s=1}^{m} f_{rs} q_r^{\cdot} q_s^{\cdot}, \quad \theta_\mu = \sum_{p=1}^{m} \theta_{\mu p} q_p^{\cdot}$$

are obtained from $2T$, $2F$ and $\partial T / \partial q_\mu^{\cdot}$ by the elimination of $q_\mu^{\cdot}$ using formulas (1.1).

We assume that the coordinates $q_\alpha$ $(\alpha = l+1, \ldots, m)$ are ignorable coordinates in the meaning of the definition in [2] which generalizes the definition in [3], i.e. coordinates $q_\alpha$ do not explicitly appear in Eqs. (1.2) where only their accelerations and, possibly, velocities are present. More exactly, we assume that

$$\frac{\partial T^*}{\partial q_\alpha} = 0, \quad \frac{\partial U}{\partial q_\alpha} = 0, \quad \frac{\partial F^*}{\partial q_\alpha} = 0, \frac{\partial}{\partial q_\alpha} \sum_{\mu=m+1}^{n} \theta_{\mu p} \nu_{\mu rs} = 0 \tag{1.3}$$

$$(\alpha = l+1, \ldots, m; \; p, r, s = 1, \ldots, m)$$

We further assume that

$$\frac{\partial F^*}{\partial q_\alpha^{\cdot}} = 0, \quad \sum_{\mu=m+1}^{n} \theta_{\mu\gamma} \nu_{\mu\alpha\beta} \equiv 0 \quad (\alpha, \beta, \gamma = l+1, \ldots, m) \tag{1.4}$$

The first group of conditions in (1.4) implies the absence of dissipation with respect to cyclic velocities and the second ensures the existence of an $(m - l)$-dimensional manifold of steady motions. Obviously, system (1.2) admits under conditions (1.3) and (1.4) the solution

$$q_i = q_{i0}, \quad q_i^{\cdot} = 0 \quad (i = 1, \ldots, l); \tag{1.5}$$

$$q_\alpha^{\cdot} = q_{\alpha 0}^{\cdot} \quad (\alpha = l+1, \ldots, m)$$

where the $m$ constants $q_{i0}$ and $q_{\alpha 0}^{\cdot}$ satisfy the system of $l < m$ equations

$$\frac{\partial U}{\partial q_i} + \sum_{\alpha, \beta = l+1}^{m} q_\alpha^{\cdot} q_\beta^{\cdot} \left[ \frac{1}{2} \frac{\partial q_{\alpha\beta}}{\partial q_i} + \sum_{\mu=m+1}^{n} \theta_{\mu\beta} \nu_{\mu i\alpha} \right] = 0 \quad (i = 1, \ldots, l) \tag{1.6}$$

Let us consider an arbitrary point of manifold (1.6) and formulate the problem of stability of solution (1.5) of system (1.2) with respect to perturbations of variables $q_i$, $q_i^{\cdot}$ and $q_\alpha^{\cdot}$.

2. We set

$$x_i = q_i - q_{i0} \quad (i = 1, \ldots, l), \quad y_\alpha = q_\alpha^{\cdot} - \omega_\alpha \quad (\omega_\alpha = q_{\alpha 0}^{\cdot},$$

$$\alpha = l+1, \ldots, m)$$

and write the equations of perturbed motion as

$$\sum_j a_{ij} x_j^{\cdot\cdot} + \sum_\beta a_{i\beta} y_\beta^{\cdot} = \sum_{j,h} x_j^{\cdot} x_h^{\cdot} B_{ijh} + \sum_{j,\beta} x_j^{\cdot} (\omega_\beta + y_\beta) B_{ij\beta} + \tag{2.1}$$

$$\sum_{\beta,\gamma} \omega_\beta \omega_\gamma \Delta B_{i\beta\gamma} + \sum_{\beta,\gamma} (\omega_\beta y_\gamma + \omega_\gamma y_\beta + y_\beta y_\gamma) B_{i\beta\gamma} +$$

$$\Delta \frac{\partial U}{\partial q_i} - \sum_j f_{ij} x_j^{\cdot}$$

$$\sum_j a_{\alpha j} x_j^{\cdot\cdot} + \sum_\beta a_{\alpha\beta} y_\beta^{\cdot} = \sum_{j,h} x_j^{\cdot} x_h^{\cdot} B_{\alpha jh} + \sum_{j,\beta} x_j^{\cdot} (\omega_\beta + y_\beta) B_{\alpha j\beta}$$

in which and everywhere below $i, j, h = 1, \ldots, l$; $\alpha, \beta, \gamma = l + 1, \ldots, m$ and $\mu = m + 1, \ldots, n$. All coefficients of system (2.1) are calculated for $q_i = q_{i0} + x_i$, and symbol $\Delta$ is defined as follows:

$$\Delta\psi = \psi(q_0 + x) - \psi(q_0)$$

$$B_{ijh} = \frac{1}{2}\frac{\partial a_{jh}}{\partial q_i} - \frac{\partial a_{ij}}{\partial q_h} + \sum_\mu \theta_{\mu h}\nu_{\mu ij}$$

$$B_{ij\beta} = \frac{\partial a_{j\beta}}{\partial q_i} - \frac{\partial a_{i\beta}}{\partial q_j} + \sum_\mu (\theta_{\mu\beta}\nu_{\mu ij} + \theta_{\mu j}\nu_{\mu i\beta})$$

$$B_{i\beta\gamma} = \frac{1}{2}\frac{\partial a_{\beta\gamma}}{\partial q_i} + \sum_\mu \theta_{\mu\beta}\nu_{\mu i\gamma}$$

$$B_{\alpha jh} = \sum_\mu \theta_{\mu h}\nu_{\mu\alpha j} - \frac{\partial a_{\alpha j}}{\partial q_h}$$

$$B_{\alpha j\beta} = \sum_\mu (\theta_{\mu\beta}\nu_{\mu\alpha j} + \theta_{\mu j}\nu_{\mu\alpha\beta}) - \frac{\partial a_{\alpha\beta}}{\partial q_j}$$

The first approximation equations in the neighborhood of solution (1.5) assumes the form

$$\sum_j a_{ij}^\circ x_j^{\cdot\cdot} + \sum_\beta a_{i\beta}^\circ y_\beta^{\cdot} = \sum_j (g_{ij}^\circ + d_{ij}^\circ) x_j^{\cdot} + \sum_j (c_{ij}^\circ + e_{ij}^\circ) x_j + \sum_\beta u_{i\beta}^\circ y_\beta \qquad (2.2)$$

$$\sum_j a_{\alpha j}^\circ x_j^{\cdot\cdot} + \sum_\beta a_{\alpha\beta}^\circ y_\beta^{\cdot} = \sum_j v_{\alpha j}^\circ x_j^{\cdot}$$

$$g_{ij} + d_{ij} = \sum_\beta \omega_\beta B_{ij\beta} - f_{ij}; \quad g_{ij} = -g_{ji}, \quad d_{ij} = d_{ji}$$

$$c_{ij} + e_{ij} = \frac{\partial^2 U}{\partial q_j \partial q_i} + \sum_{\beta,\gamma} \omega_\beta\omega_\gamma \frac{\partial B_{i\beta\gamma}}{\partial q_j}; \quad c_{ij} = c_{ji}, \quad e_{ij} = -e_{ji}$$

$$u_{i\alpha} = \sum_\beta \omega_\beta (B_{i\beta\alpha} + B_{i\alpha\beta}), \quad v_{\alpha j} = \sum_\beta \omega_\beta B_{\alpha j\beta}$$

where the superscript $^\circ$ indicates that the particular quantity is calculated for $x_i = 0$ (in initial variables for $q_i = q_{i0}$).

The characteristic eqution of system (2.2) has $m - l$ zero roots with the remaining $2l$ roots satisfying the equation

$$\det \begin{Vmatrix} \| a_{ij}^\circ\lambda^2 - (g_{ij}^\circ + d_{ij}^\circ)\lambda - (c_{ij}^\circ + e_{ij}^\circ) \| & \| a_{i\beta}^\circ\lambda - u_{i\beta}^\circ \| \\ \| a_{\alpha j}^\circ\lambda - v_{\alpha j}^\circ \| & \| a_{\alpha\beta}^\circ \| \end{Vmatrix} = 0 \qquad (2.3)$$

When at least one of the roots of Eq. (2.3) lies in the right-hand half-plane, solution (1.5) is unstable. If, however, all roots of Eq. (2.3) are in the left-hand half-plane, we have conditions of the critical case of several zero roots. We shall show that the particular case of several zero roots occurs in this problem.

3. Equations (2.2) have obviously $m - l$ linear integrals

$$\sum_j a_{\alpha j}{}^\circ x_j^{\boldsymbol\cdot} + \sum_\beta \mathring{a}_{\alpha\beta} y_\beta - \sum_{j,\beta} x_j \omega_\beta \mathring{B}_{\alpha j\beta} = z_\alpha \quad (z_\alpha = \text{const}, \alpha = l+1, \ldots, m) \tag{3.1}$$

We substitute variables $z$ defined by formulas (3.1) for variables $y$, and write down the system of equations of perturbed motion in variables $x$, $x^{\boldsymbol\cdot}$, and $z$ after resolving system (2.1) for higher derivatives. We then obtain

$$\begin{aligned} x_i^{\boldsymbol\cdot} &= x_i' \\ x_i'^{\boldsymbol\cdot} &= \sum_j A_{ij}(x)\,\Phi_j(x,z) + \sum_j x_j' \Psi_{ij}(x,x',z) \\ z_\alpha^{\boldsymbol\cdot} &= \sum_j \Big(\sum_s \mathring{a}_{\alpha s} A_{sj}(x)\Big)\Phi_j(x,z) + \sum_j x_j' \Psi_{\alpha j}(x,x',z) \end{aligned} \tag{3.2}$$

where $A_{rs}$ are elements of the matrix inverse of matrix $\| a_{rs} \|$ $(r, s, = 1, \ldots, m)$; expansion of functions $\Phi_i$ in powers of $x$ and $z$, and of functions $\Psi_{\alpha j}$ in powers of $x$, $x'$, and $z$, beginning with terms of order not lower than the first, and functions $\Psi_{ij}{}^\circ$ are generally not equal zero. Functions $\Phi_i$, $\Psi_{ij}$ and $\Psi_{\alpha j}$ are not presented explicitly owing to their unwieldiness. Note that expansions of the right-hand sides of the last $m - l$ equations of system (3.2) begins with terms of order not lower than the second, since

$$\sum_s \mathring{a}_{\alpha s} A_{sj}{}^\circ = \delta_{\alpha j} = 0 \quad (\alpha \neq j)$$

Since the expansion of functions $\Phi_i(x, z)$ may contain terms that are linear with respect to $z$, it is necessary to carry out the transformation of variables, which would reduce the system to the form that is standard for the analysis of the critical case of several zero roots. For this we consider the system of equations

$$x_i' = 0, \sum_j A_{ij}(x)\,\Phi_j(x,z) + \sum_j x_j' \Psi_{ij}(x,x',z) = 0$$

whose solution for $x$ and $x'$ yields

$$x_i' = 0, \quad x_i = X_i(z)$$

where functions $X_i$ satisfy the system of equations

$$\Phi_i(X, z) = 0 \quad (\det \| A_{ij} \| \neq 0)$$

whose solution is a priori known to exist, since by assumption all roots of Eq. (2.3) lie in the left-hand half-plane.

We carry out the change of variables

$$x_i = X_i(z) + u_i, \quad x_i' = v_i$$

and write down the system of equations of perturbed motion in variables $u, v, z$. We have

$$\begin{aligned} u_i^{\cdot} &= L_i(u, v) + U_i(u, v, z) \\ v_i^{\cdot} &= L_{l+i}(u, v) + V_i(u, v, z) \\ z_\alpha^{\cdot} &= \sum_j \Big[ \sum_s \overset{\circ}{a}_{\alpha s} A_{sj}(X(z) + u) \Big] \Phi_j(X(z) + u) + \\ &\quad \sum_j v_j \Psi_{\alpha j}(X(z) + u, v, z) \end{aligned} \tag{3.3}$$

where $L_k(u, v)$ $(k = 1, \ldots, 2l)$ are linear forms of variables $u$ and $v$, and the expansions of functions $U_i$ and $V_i$ in powers of $u, v$, and $z$ begin with terms of order not lower than the second.

When $u = v = 0$ then obviously the right-hand sides of the last $m - l$ equations of system (3.3) are identically zero, consequently, also $U_i(0, 0, z) \equiv V_i(0, 0, z) \equiv 0$ [4, 5].

Thus the statement that when all roots of Eq. (2.3) lie in the left-hand half-plane, the particular case of the critical case of several zero roots is realized, is proved. Consequently, when all roots of Eq. (2.3) are in th left-hand half-plane, then solution (1.5) of system (1.2) is stable (but not asymptotically). Any perturbed motion fairly close to the unperturbed tends to one of the possible steady (but not to the unperturbed) motions of the form (1.5) that belong to the manifold (1.6) when $t \to \infty$.

4. Equation (2.3) can be reduced by elementary transformations to the form

$$\det \| A\lambda^2 - (G + D)\lambda - (C + E) \| = 0 \tag{4.1}$$

which is the characteristic equation for the system

$$\begin{aligned} &Aw^{\cdot\cdot} = Gw^{\cdot} + Dw^{\cdot} + Cw + Ew \\ &w = \operatorname{colon}(w_1, \ldots, w_l), \quad A = \| a_{ij}^\circ - \sum_{\alpha, \beta} a_{i\beta}^\circ \overset{\circ}{h}_{\beta\alpha} a_{j\alpha}^\circ \| \\ &G + D = \| g_{ij}^\circ + d_{ij}^\circ - \sum_{\alpha, \beta} h_{\alpha\beta} (a_{i\beta}^\circ v_{j\alpha}^\circ + a_{j\alpha}^\circ u_{i\beta}^\circ) \| \\ &G' = -G, \quad D' = D \\ &C + E = \| c_{ij}^\circ + e_{ij}^\circ + \sum_{\alpha, \beta} \overset{\circ}{h}_{\alpha\beta} u_{i\beta}^\circ v_{j\alpha}^\circ \|, \quad C' = C, \quad E' = -E \end{aligned} \tag{4.2}$$

where $h_{\alpha\beta}^{\circ}$ are elements of the matrix inverse of matrix $\| a_{\alpha\beta}^{\circ} \|$ $(\alpha, \beta = l + 1, \ldots, m)$ and the prime indicates transposition.

Matrix $A$ is evidently of positive definite quadratic form, i.e. it is possible to consider system (4.2) as consisting of equations of motion of a mechanical system subjected to the following types of forces: potential $Cw$, positional nonconservative $Ew$, gyroscopic $Gw^{\cdot}$, and dissipative and accelerating $Dw^{\cdot}$. The following statement is valid on the basis of the above exposition.

The steady motion (1.5) of system (1.2) is stable (unstable) with respect to variables $q_i - q_{i0}$, $q_i^{\cdot}$, and $q_{\alpha}^{\cdot} - q_{\alpha 0}^{\cdot}$ when the zero equilibrium position of system (4.2) is asymptotically stable (exponentially unstable).

A number os theorems on stability or instability of steady motions of Chaplygin's nonholonomic systems can be obtained using the last statement and results of investigations of systems of the form (4.2) [6 — 10], as was done earlier in investiagion of the stability of equilibrium positions of nonholonomic systems [11].

When $G \equiv 0$ all theorems in [11] are valid, and when $G \not\equiv 0$ and $E \equiv 0$ Theorems 3.1, 3.3, and 3.4 in [11] are valid, while Theorem 3.2 is not. Finally, when $G \not\equiv 0$ and $E \not\equiv 0$, the following statements are valid.

1°. If function $2V = -w'Cw$ has a minimum at the coördinate origin and $D = -\delta D_*$, where $D_*$ is the matrix of positive definite form, then for fairly large $\delta > 0$ the steady motion (1.5) of system (1.2) is stable.

2°. When one of the following conditions is satisfied:

a) $D \equiv 0$ Eq. (4.1) contains odd powers of $\lambda$;

b) $\det \| -(C + E) \| < 0$; and

c) function $2H_0 = -w' (^1/_4 GA^{-1}G + C)w$ has a maximum at the coordinate origin, the steady motion (1.5) of system (1.2) is unstable.

R e m a r k. Since $(q_{i0}, q_{\alpha 0}^{\cdot})$ is an arbitrary point of the manifold of steady motions (1.6), the obtained results make possible the investigation of all motions of the input system (since all coefficients of system (4.2) depend on $q_{i0}$, and $q_{\alpha 0}^{\cdot}$). If the solution of system (1.6) can be represented in parametric form, for instance, in the form

$$q_{\alpha 0}^{\cdot} = \omega_{\alpha} \quad (\alpha = l + 1, \ldots, m); \quad q_{i0} = \varphi_i(\omega) \quad (i = 1, \ldots, l) \qquad (4.3)$$

then, by substituting (4.3) into the formulas for coefficients of matrices of system (4.2) and using the obtained results it is possible to separate on surface (4.3) regions of stable or unstable steady motions.

5. We illustrate the above results on the example of investigation of stability of steady motion of a torus on an absolutely rough horizontal surface.

We define the motions of the torus by the Cartesian coordinates $x$ and $y$ of the center of mass projection on the horizontal plane and by Euler's angles $\theta, \psi$, and $\varphi$. We denote by $m$ the torus mass, by $A$ and $B$ the equatorial and polar moments of inertia, by $r$ the radus of the torus cross section, and by $R + r$ the radius of the equatorial circle. In that notation the Lagrange function and the system relation equations that define the absence of slip at the point of torus contact with the plane are of the form

$$L = {}^1/_2 m\,(x^{\cdot 2} + y^{\cdot 2}) + {}^1/_2\,(A + mR^2 \sin^2\theta)\theta^{\cdot 2} + {}^1/_2\,(A \cos^2\theta + B \sin^2\theta)\psi^{\cdot 2} + {}^1/_2 B\varphi^{\cdot 2} - B\varphi^\cdot\psi^\cdot \sin\theta - mgR\cos\theta$$

$$x^\cdot = (R + r\cos\theta)\cos\psi\varphi^\cdot - R\sin\theta\cos\psi\psi^\cdot - (R\cos\theta + r)\sin\psi\theta^\cdot$$

$$y^\cdot = (R + r\cos\theta)\sin\psi\varphi^\cdot - R\sin\theta\sin\psi\psi^\cdot + (R\cos\theta + r)\cos\psi\theta^\cdot.$$

Assuming that the system can be subjected to the action of dissipative forces, derivatives of the Rayleigh function $F = {}^1/_2 H\theta^{\cdot 2}$ ($H = \text{const} > 0$), we can readily show that $\psi$ and $\varphi$ are ignorable coordinates in the sense of the definition (1.3) and (1.4). Hence the input system can perform steady motions of the form

$$\theta = \alpha, \quad \varphi^\cdot = \omega, \quad \psi^\cdot = \Omega \tag{5.1}$$

with the three constants $\alpha$, $\omega$ and $\Omega$ satisfying the single equation

$$mgR\sin\alpha - [B\cos\alpha + m\,(R\cos\alpha + r)(R + r\cos\alpha)]\omega\Omega + [(B - A)\cos\alpha + mR\,(R\cos\alpha + r)]\sin\alpha\Omega^2 = 0 \tag{5.2}$$

Applying these results to an arbitrary point of the manifold (5.2) we find that the steady motion (5.1) is stable (unstable), if the trivial solution of equation

$$[A + m\,(R^2 + 2Rr\cos\alpha + r^2)]w^{\cdot\cdot} + Hw^\cdot + J\,(K\Omega^2 + L\Omega\omega + M\omega^2 + N)w = 0 \tag{5.3}$$

$$J = [AB + Am\,(R + r\cos\alpha)^2 + Bmr^2\sin^2\alpha]^{-1}\cos^{-1}\alpha > 0$$

$$K = [B\cos\alpha + m\,(R\cos\alpha + r)(R + r\cos\alpha)][Am\,(R + r\cos\alpha)(2R + r\cos\alpha) + AB\,(1 + \sin^2\alpha) + Bm\sin^2\alpha\,(r^2 - R^2) - B^2\sin^2\alpha] - [(B - A + mR^2)\cos 2\alpha + mRr\cos\alpha][AB + Am\,(R + r\cos\alpha)^2 + Bmr^2\sin^2\alpha]\cos\alpha - 2\,[(B - A + mR^2)\cos\alpha + mRr][-B^2 + 2AB + 2Am\,(R + r\cos\alpha)^2 - Bm\,(R^2 + Rr\cos\alpha + r^2\cos 2\alpha)]\sin^2\alpha$$

$$L = [B\cos\alpha + m\,(R\cos\alpha + r)(R + r\cos\alpha)][Am\,(2R^2 + 5Rr\cos\alpha + 3r^2\cos^2\alpha) + AB + Bmr^2\,(3\sin^2\alpha - 1)]\sin\alpha - [B + m\,(R^2 + 2Rr\cos\alpha + r^2)][-AB\cos\alpha + Am\,(R + r\cos\alpha)^2\cos\alpha + 2B^2\cos\alpha + Bm\,(2R^2\cos\alpha + 2Rr + r^2\sin^2\alpha\cos\alpha)]\sin\alpha,\quad M = B\,[B + m\,(R^2 + 2Rr\cos\alpha + r^2)][B\cos\alpha + m\,(R\cos\alpha + r)(R + r\cos\alpha)]$$

$$N = -mgR\,[AB + Am\,(R + r\cos\alpha)^2 + Bmr^2\sin^2\alpha]\cos^2\alpha$$

is asymptotically stable (exponentially unstable). This is the equation of motion of a mechanical system with a single degree of freedom subjected to the action of dissipative and potential forces. From this we immediately obtain the condition of stability (instability)

$$K\Omega^2 + L\Omega\omega + M\omega^2 + N > 0 \quad (< 0) \tag{5.4}$$

of steady motion (5.1) in the form of the minimum (absence of minimum) of potential energy of system (5.3).

When $r = 0$ condition (5.4) becomes the condition of stability of the steady motion of a hoop (see, e.g., [1,3]), when $\alpha = 0$ and $\omega = 0$ it becomes the condition of stability of a torus spinning about the vertical at constant angular velocity, and when $\alpha = 0$ and $\Omega = 0$ it becomes the condition of stability of uniform rolling of a torus along a straight line (see, e.g., [3]).

## REFERENCES


1. Rumiantsev, V. V. and Karapetian, A. V., Stability of motion of nonholonomic systems. Attainments of Science and Technology: General Mechanics, Vol. 3, Moscow, VINITI, 1976.
2. Emel'ianova, I. S. and Fufaev, N. A., On the stability of steady motions. Collection: Theory of Oscillations, Applied Mathematics and Cybernetics. Izd. Gor'kovsk. Univ., 1974.
3. Neimark, Iu. I. and Fufaev, N. A., Dynamics of Nonholonomic Systems. Moscow, "Nauka", 1967.
4. Liapunov, A. M., Collected Works, Vol. 2, Moscow—Leningrad, Izd. Akad. Nauk. SSSR, 1956.
5. Malkin, I. G., Theory of Motion Stability. Moscow, "Nauka", 1966.
6. Metelitsyn, I. I., The problem of gyroscopic stabilization. Dokl. Akad. Nauk. Vol. 86, No. 1, 1952.
7. Merkin, D. R., Introduction to the Theory of Motion Stability. Moscow, "Nauka", 1971.
8. Lakhadanov, V. M., On the influence of structure of forces on the stability of motion. PMM, Vol. 38, No. 2, 1974.
9. Lakhadanov, V. M., On stabilization of potential systems. PMM, Vol. 39, No. 1, 1975.
10. Karapetian, A. V., On the stability of nonconservative systems. Vestn. MGU, Ser. Matem. Mekhan., No. 4, 1975.
11. Karapetian, A. V., On stability of equilibrium of nonholonomic systems. PMM, Vol. 39, No. 6, 1975.


Translated by J.J.D.